

# L-functions

(PARI-GP version 2.15.5)

## Characters

A character on the abelian group  $G = \sum_{j \leq k} (\mathbf{Z}/d_j \mathbf{Z}) \cdot g_j$ , e.g. from **znstar**(**q**,1)  $\leftrightarrow (\mathbf{Z}/q \mathbf{Z})^*$  or **bnrinit**  $\leftrightarrow \text{Cl}_{\mathbb{F}}(K)$ , is coded by  $\chi = [c_1, \dots, c_k]$  such that  $\chi(g_j) = e(c_j/d_j)$ . Our  $L$ -functions consider the attached *primitive* character.

Dirichlet characters  $\chi_q(m, \cdot)$  in Conrey labelling system are alternatively concisely coded by **Mod**(**m**,**q**). Finally, a quadratic character ( $D/\cdot$ ) can also be coded by the integer  $D$ .

## L-function Constructors

An **Ldata** is a GP structure describing the functional equation for  $L(s) = \sum_{n \geq 1} a_n n^{-s}$ .

- Dirichlet coefficients given by closure  $a : N \mapsto [a_1, \dots, a_N]$ .
- Dirichlet coefficients  $a^*(n)$  for dual  $L$ -function  $L^*$ .
- Euler factor  $A = [a_1, \dots, a_d]$  for  $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$ ,
- classical weight  $k$  (values at  $s$  and  $k - s$  are related),
- conductor  $N$ ,  $\Lambda(s) = N^{s/2} \gamma_A(s)$ ,
- root number  $\varepsilon$ ;  $\Lambda(a, k - s) = \varepsilon \Lambda(a^*, s)$ .
- polar part: list of  $[\beta, P_{\beta}(x)]$ .

An **Linit** is a GP structure containing an **Ldata**  $L$  and an evaluation *domain* fixing a maximal order of derivation  $m$  and bit accuracy (**realbitprecision**), together with complex ranges

- for  $L$ -function:  $R = [c, w, h]$  (coding  $|\Re z - c| \leq w$ ,  $|\Im z| \leq h$ ); or  $R = [w, h]$  (for  $c = k/2$ ); or  $R = [h]$  (for  $c = k/2$ ,  $w = 0$ ).
- for  $\theta$ -function:  $T = [\rho, \alpha]$  (for  $|t| \geq \rho$ ,  $|\arg t| \leq \alpha$ ); or  $T = \rho$  (for  $\alpha = 0$ ).

### Ldata constructors

|   |   |
|---|---|
| Riemann $\zeta$   | <b>lfuncreate</b> (1)   |
| Dirichlet for quadratic char. ( $D/\cdot$ )                               | <b>lfuncreate</b> ( $D$ )   |
| Dirichlet series $L(\chi_q(m, \cdot), s)$                                 | <b>lfuncreate</b> ( <b>Mod</b> ( <b>m</b> , <b>q</b> ))           |
| Dedekind $\zeta_K$ , $K = \mathbf{Q}[x]/(T)$                              | <b>lfuncreate</b> ( <i>bnf</i> ), <b>lfuncreate</b> ( $T$ )       |
| Hecke for $\chi \bmod \mathfrak{f}$                                       | <b>lfuncreate</b> ( <i>bnr</i> , $\chi$ )                         |
| Artin $L$ -function   | <b>lfunartin</b> ( <i>nf</i> , <i>gal</i> , $M$ , $n$ )           |
| Lattice $\Theta$ -function  | <b>lfunqf</b> ( $Q$ )   |
| From eigenform $F$  | <b>lfunmf</b> ( $F$ )   |
| Quotients of Dedekind $\eta : \prod_i \eta(m_{i,1} \cdot \tau)^{m_{i,2}}$ | <b>lfunetaquo</b> ( $M$ )   |
| $L(E, s)$ , $E$ elliptic curve  | <b>E = ellinit</b> (...)  |
| $L(\text{Sym}^m E, s)$ , $E$ elliptic curve                               | <b>lfunsympow</b> ( <b>E</b> , <b>m</b> )                         |
| Genus 2 curve, $y^2 + xQ = P$   | <b>lfungenus2</b> ( $[P, Q]$ )                                    |
| Hypergeometric motive $H(a, b; t)$  | <b>lfunhgm</b> ( <b>hgmin</b> ( <b>a</b> , <b>b</b> ), <b>t</b> ) |

|                              |                               |
|------------------------------|-------------------------------|
| dual $L$ function $\hat{L}$  | <b>lfundual</b> ( $L$ )       |
| $L_1 \cdot L_2$              | <b>lfunmul</b> ( $L_1, L_2$ ) |
| $L_1/L_2$                    | <b>fundiv</b> ( $L_1, L_2$ )  |
| $L(s - d)$                   | <b>funshift</b> ( $L, d$ )    |
| $L(s) \cdot L(s - d)$        | <b>funshift</b> ( $L, d, 1$ ) |
| twist by Dirichlet character | <b>funtwist</b> ( $L, \chi$ ) |

|                                     |   |
|-------------------------------------|---|
| low-level constructor               | <b>lfuncreate</b> ( $[a, a^*, A, k, N, \textit{eps}, \textit{poles}]$ ) |
| check functional equation (at $t$ ) | <b>funcheckfeq</b> ( $L, \{t\}$ )                                       |
| parameters $[N, k, A]$              | <b>funparams</b> ( $L$ )  |

### Linit constructors

|  |  |
|--|--|
| initialize for $L$                               | <b>lfuninit</b> ( $L, R, \{m = 0\}$ )              |
| initialize for $\theta$                          | <b>lfunthetainit</b> ( $L, \{T = 1\}, \{m = 0\}$ ) |
| cost of <b>lfuninit</b>                          | <b>funcost</b> ( $L, R, \{m = 0\}$ )               |
| cost of <b>lfunthetainit</b>                     | <b>funthetacost</b> ( $L, T, \{m = 0\}$ )          |
| Dedekind $\zeta_L$ , $L$ abelian over a subfield | <b>funabelianreinit</b>                            |

## L-functions

$L$  is an **Ldata** or an **Linit** (more efficient for many values).

### Evaluate

|                    |   |
|--------------------|---|
| $L^{(k)}(s)$       | <b>lfun</b> ( $L, s, \{k = 0\}$ )       |
| $\Lambda^{(k)}(s)$ | <b>lfunlambda</b> ( $L, s, \{k = 0\}$ ) |
| $\theta^{(k)}(t)$  | <b>lfuntheta</b> ( $L, t, \{k = 0\}$ )  |

generalized Hardy  $Z$ -function at  $t$  **lfunhardy**( $L, t$ )

### Zeros

|  |   |
|--|---|
| order of zero at $s = k/2$               | <b>funorderzero</b> ( $L, \{m = -1\}$ ) |
| zeros $s = k/2 + it$ , $0 \leq t \leq T$ | <b>funzeros</b> ( $L, T, \{h\}$ )       |

### Dirichlet series and functional equation

|                           |                             |
|---------------------------|-----------------------------|
| $[a_n : 1 \leq n \leq N]$ | <b>funan</b> ( $L, N$ )     |
| Euler factor at $p$       | <b>fun euler</b> ( $L, p$ ) |
| conductor $N$ of $L$      | <b>funconductor</b> ( $L$ ) |
| root number and residues  | <b>funrootres</b> ( $L$ )   |

### G-functions

Attached to inverse Mellin transform for  $\gamma_A(s)$ ,  $A = [a_1, \dots, a_d]$ .

|                                    |  |
|------------------------------------|--|
| initialize for $G$ attached to $A$ | <b>gammamellinin</b> ( $A$ )                   |
| $G^{(k)}(t)$                       | <b>gammamellinin</b> ( $G, t, \{k = 0\}$ )     |
| asyp. expansion of $G^{(k)}(t)$    | <b>gammamellinvasymp</b> ( $A, n, \{k = 0\}$ ) |

## Hypergeometric motives (HGM)

### Hypergeometric templates

Below,  $H$  denotes an hypergeometric template from **hgminit**.  
HGM template from  $A = (\alpha_j), B = (\beta_k)$  **hgminit**( $A, \{B\}$ )  
...from cyclotomic parameters  $D, E$  **hgminit**( $D, \{E\}$ )  
...from gamma vector **hgminit**( $G$ )  
 $\alpha$  and  $\beta$  parameters for  $H$  **hgmalph**( $H$ )  
cyclotomic parameters ( $D, E$ ) of  $H$  **hgmcyclo**( $H$ )  
...for all  $H$  of degree  $n$  **hgmbdegree**( $n$ )  
gamma vector for  $H$  **hgmgamma**( $H$ )  
twist  $A$  and  $B$  by  $1/2$  **hgmtwist**( $H$ )  
is  $H$  symmetrical at  $t = 1$ ? **hgmissymmetrical**( $H$ )  
parameters  $[d, w, [P, T], M]$  for  $H$  **hgmparams**( $H$ )

### L-function

Let  $L$  be the  $L$ -function attached to the hypergeometric motive ( $H, t$ ).

|   |  |
|---|--|
| coefficient $a_n$ of $L$                | <b>hgcoef</b> ( $H, t, n$ )              |
| coefficients $[a_1, \dots, a_n]$ of $L$ | <b>hgcoefs</b> ( $H, t, n$ )             |
| Euler factor at $p$                     | <b>hgmeulerfactor</b> ( $H, t, p$ )      |
| ...and valuation of local conductor     | <b>hgmeulerfactor</b> ( $H, t, p, \&e$ ) |
| return $L$ as an <b>Ldata</b>           | <b>lfunhgm</b> ( $H, t$ )                |

Based on an earlier version by Joseph H. Silverman

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